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ON SINGULAR CUBIC SURFACES

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ABSTRACT. We study global log canonical thresholds of cubic surfaces with canonical singularities, and we prove the existence of a Kähler–Einstein metric on two singular cubic surfaces.

1. INTRODUCTION.

Let X be a Fano variety¹ with log terminal singularities, and G be a finite subgroup in $\text{Aut}(X)$.

Definition 1.1. Global G -invariant log canonical threshold of the variety X is the number

$$\text{lct}(X, G) = \sup \left\{ \lambda \in \mathbb{Q} \mid \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ has log canonical singularities} \\ \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \equiv -K_X \end{array} \right\}.$$

We put $\text{lct}(X) = \text{lct}(X, G)$ in the case when G is a trivial group.

Example 1.2. Let X be a smooth hypersurface in \mathbb{P}^n of degree n . Then $\text{lct}(X) \geq 1 - 1/n$ by [2].

Example 1.3. The simple group $\text{PGL}(2, \mathbb{F}_7)$ is a group of automorphisms of the quartic curve

$$x^3y + y^3z + z^3x = 0 \subseteq \mathbb{P}^2 \cong \text{Proj}(\mathbb{C}[x, y, z]),$$

which induces an embedding $\text{PGL}(2, \mathbb{F}_7) \subseteq \text{Aut}(\mathbb{P}^2)$. Then $\text{lct}(\mathbb{P}^2, \text{PGL}(2, \mathbb{F}_7)) = 4/3$ by [5].

The number $\text{lct}(X, G)$ plays an important role in birational geometry (see Section 5).

Example 1.4. Let X be a general quasismooth hypersurface in $\mathbb{P}(1, a_1, \dots, a_4)$ of degree $\sum_{i=1}^4 a_i$ with terminal singularities such that $-K_X^3 \leq 1$. Then $\text{lct}(X) = 1$ by [4], which implies that

$$\text{Bir}\left(\underbrace{X \times \cdots \times X}_{m \text{ times}}\right) = \left\langle \prod_{i=1}^m \text{Bir}(X), \text{Aut}\left(\underbrace{X \times \cdots \times X}_{m \text{ times}}\right) \right\rangle,$$

and the variety $X \times \cdots \times X$ is non-rational (see [7], [15], [4]).

The number $\text{lct}(X, G)$ plays an important role in Kähler geometry.

Example 1.5. Suppose that X has at most quotient singularities, and the inequality

$$\text{lct}(X, G) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then X has a Kähler–Einstein metric (see [8]).

Let S be a del Pezzo surface with canonical singularities. Put $\Sigma = \text{Sing}(S)$.

Remark 1.6. It follows from [16], [9], [13], [11], [10], [5] that

- the surface S has a Kähler–Einstein metric in the following cases:
 - when $\Sigma = \emptyset$, $S \not\cong \mathbb{F}_1$ and $K_S^2 \neq 7$;
 - when S is a complete intersection

$$\sum_{i=0}^4 x_i^2 = \sum_{i=0}^4 \lambda_i x_i^2 = 0 \subseteq \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_4]),$$

and Σ consists of points of type \mathbb{A}_1 , where $\lambda_i \in \mathbb{C}$;

- when $K_S^2 = 2$, and Σ consists of points of types \mathbb{A}_1 and \mathbb{A}_2 ;

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¹We assume that all varieties are projective, normal, and defined over \mathbb{C} .

- when $K_S^2 = 1$, and Σ consists of points of type \mathbb{A}_1 ;
- the surface S does not have a Kähler–Einstein metric in the following cases:
 - when $\Sigma = \emptyset$, and either $S \cong \mathbb{F}_1$ or $K_S^2 = 7$;
 - when Σ contains a point that is not of type \mathbb{A}_1 , and $K_S^2 = 4$;
 - when Σ contains a point that is not of type \mathbb{A}_1 or \mathbb{A}_2 , and $K_S^2 = 3$.

All possible values of $\text{lct}(S)$ are found in [5] in the case when $\Sigma = \emptyset$.

Example 1.7. Suppose that S is a cubic surface in \mathbb{P}^3 and $\Sigma = \emptyset$. Then

$$\text{lct}(S) = \begin{cases} 2/3 & \text{when } S \text{ has an Eckardt point,} \\ 3/4 & \text{when } S \text{ does not have Eckardt points.} \end{cases}$$

We prove the following result in Sections 3.

Theorem 1.8. *Suppose that S is a cubic surface in \mathbb{P}^3 and $\Sigma \neq \emptyset$. Then*

$$\text{lct}(S) = \begin{cases} 2/3 & \text{when } \Sigma = \{\mathbb{A}_1\}, \\ 1/3 & \text{when } \Sigma \supseteq \{\mathbb{A}_4\}, \\ 1/3 & \text{when } \Sigma = \{\mathbb{D}_4\}, \\ 1/3 & \text{when } \Sigma \supseteq \{\mathbb{A}_2, \mathbb{A}_2\}, \\ 1/4 & \text{when } \Sigma \supseteq \{\mathbb{A}_5\}, \\ 1/4 & \text{when } \Sigma = \{\mathbb{D}_5\}, \\ 1/6 & \text{when } \Sigma = \{\mathbb{E}_6\}, \\ 1/2 & \text{in other cases.} \end{cases}$$

The group S_4 naturally acts on the cubic surface $\dot{S} \subset \mathbb{P}^3$ that is given by the equation

$$(1.9) \quad xyz + xyt + xzt + yzt = 0 \subseteq \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, t]),$$

the group $S_3 \times \mathbb{Z}_3$ naturally acts on the cubic surface $\dot{S} \subset \mathbb{P}^3$ that is given by the equation

$$(1.10) \quad xyz = t^3 \subseteq \mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, t]),$$

and $\text{lct}(\dot{S}, S_4) = \text{lct}(\dot{S}, S_3 \times \mathbb{Z}_3) = 1$ (see Section 4). But both surfaces \dot{S} and \dot{S} are singular.

Corollary 1.11. *The surfaces \dot{S} and \dot{S} have Kähler–Einstein metrics.*

It is very likely that the method in [16] can be applied to prove the existence of a Kähler–Einstein metric on every singular cubic surface having only singular points of type \mathbb{A}_1 and \mathbb{A}_2 .

2. BASIC TOOLS.

Let S be a surface with canonical singularities, and D be an effective \mathbb{Q} -divisor on it.

Remark 2.1. Let B be an effective \mathbb{Q} -divisor on S such that (S, B) is log canonical. Then

$$\left(S, \frac{1}{1-\alpha}(D - \alpha B) \right)$$

is not log canonical if (S, D) is not log canonical, where $\alpha \in \mathbb{Q}$ such that $0 \leq \alpha < 1$.

Let $\text{LCS}(S, D) \subset S$ be a subset such that $P \in \text{LCS}(S, D)$ if and only if (S, D) is not log terminal at the point P . The set $\text{LCS}(S, D)$ is called the locus of log canonical singularities.

Remark 2.2. The set $\text{LCS}(S, D)$ is connected if $-(K_S + D)$ is ample (see Theorem 17.4 in [12]).

Let P be a point of the surface S such that (S, D) is not log canonical at the point P .

Remark 2.3. Suppose that S is smooth at P . Then $\text{mult}_P(D) > 1$.

Let C be an irreducible curve on the surface S . Put

$$D = mC + \Omega,$$

where $m \in \mathbb{Q}$ such that $m \geq 0$, and Ω is an effective \mathbb{Q} -divisor such that $C \not\subseteq \text{Supp}(\Omega)$.

Remark 2.4. Suppose that $C \subseteq \text{LCS}(S, D)$. Then $m \geq 1$.

Suppose that C is smooth at P , the inequality $m \leq 1$ holds and $P \in C$.

Remark 2.5. Suppose that S is smooth at P . Then it follows from Theorem 17.6 in [12] that

$$C \cdot \Omega \geq \text{mult}_P(\Omega|_C) > 1.$$

Let $\pi: \bar{S} \rightarrow S$ be a birational morphism, and \bar{D} is a proper transform of D via π . Then

$$K_{\bar{S}} + \bar{D} + \sum_{i=1}^r a_i E_i \equiv \pi^*(K_S + D),$$

where E_i is a π -exceptional curve, and a_i is a rational number.

Remark 2.6. The log pair (S, D) is log canonical if and only if $(\bar{S}, \bar{D} + \sum_{i=1}^r a_i E_i)$ is log canonical.

Suppose that $r = 1$, $\pi(E_1) = P$, and P is a singular point of the surface S of type \mathbb{A}_n .

Remark 2.7. Suppose that $n = 1$, and \bar{S} is smooth along E_1 . Then $a_1 > 1/2$.

Suppose that $n > 1$, and $E_1 \cap \text{Sing}(\bar{S})$ consists of one singular point of type \mathbb{A}_{n-1} .

Remark 2.8. It follows from Theorem 17.6 in [12] that $a_1 > 1/(n+1)$.

Most of the described results are valid in much more general settings (see [12]).

3. MAIN RESULT.

Let us use the assumptions and notations of Theorem 1.8. Put

$$\omega = \begin{cases} 2/3 & \text{when } \Sigma = \{\mathbb{A}_1\}, \\ 1/3 & \text{when } \Sigma \supseteq \{\mathbb{A}_4\}, \\ 1/3 & \text{when } \Sigma = \{\mathbb{D}_4\}, \\ 1/3 & \text{when } \Sigma \supseteq \{\mathbb{A}_2, \mathbb{A}_2\}, \\ 1/4 & \text{when } \Sigma \supseteq \{\mathbb{A}_5\}, \\ 1/4 & \text{when } \Sigma = \{\mathbb{D}_5\}, \\ 1/6 & \text{when } \Sigma = \{\mathbb{E}_6\}, \\ 1/2 & \text{in other cases.} \end{cases}$$

Remark 3.1. It follows from [17] that

$$w = \sup \left\{ \mu \in \mathbb{Q} \mid \text{the log pair } (S, \mu D) \text{ is log canonical for every } D \in |-K_X| \right\}.$$

Take $\lambda \in \mathbb{Q}$ such that $\lambda < \omega$. Let D be any effective \mathbb{Q} -divisor on S such that $D \equiv -K_S$.

Lemma 3.2. *Suppose that $\lambda < 1/3$. Then $\text{LCS}(S, \lambda D) \subseteq \Sigma$.*

Proof. Suppose that $(S, \lambda D)$ is not log terminal at a smooth point $P \in S$. Then

$$3 = -K_S \cdot D \geq \text{mult}_P(D) > 1/\lambda > 3,$$

which is a contradiction. □

Lemma 3.3. *Suppose that $|\text{LCS}(S, \lambda D)| < +\infty$. Then $\text{LCS}(S, \lambda D) \subseteq \Sigma$.*

Proof. The necessary assertion follows from [2] or [5]. □

Let O be the worst singular point of the surface S , and $\alpha: \bar{S} \rightarrow S$ be a partial resolution of singularities that contracts smooth rational curves E_1, \dots, E_k to the point O such that

$$\bar{S} \setminus \left(\bigcup_{i=1}^k E_i \right) \cong S \setminus O,$$

the surface \bar{S} is smooth along $\bigcup_{i=1}^k E_i$, and $E_i^2 = -2$ for every $i = 1, \dots, k$. Then

$$\bar{D} \equiv \alpha^*(D) - \sum_{i=1}^k a_i E_i,$$

where \bar{D} is the proper transform of D on the surface \bar{S} , and $a_i \in \mathbb{Q}$. Let L_1, \dots, L_r be lines on the surface S such that $O \in L_i$, and \bar{L}_i be the proper transform of L_i on the surface \bar{S} .

Lemma 3.4. *Suppose that $\Sigma = \{\mathbb{A}_1\}$. Then $\text{lct}(S) = 2/3$.*

Proof. There is a conic $C_i \subset S$ such that the singularities of the log pair $(S, \frac{2}{3}(L_i + C_i))$ are log canonical and not log terminal. So, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there is an irreducible curve $Z \subset S$ such that $D = \mu Z + \Omega$, where μ is a rational number such that $\mu \geq 1/\lambda$, and Ω is an effective \mathbb{Q} -divisor such that $Z \not\subset \text{Supp}(S)$. Then

$$3 = -K_S \cdot D = \mu \deg(Z) - K_S \cdot \Omega \geq \mu \deg(Z) > 3 \deg(Z)/2,$$

which implies that Z is a line. Let C be a general conic on S such that $-K_S \sim Z + C$. Then

$$2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geq \mu C \cdot Z \geq \frac{3}{2}\mu,$$

which is a contradiction. Then $\text{LCS}(S, \lambda D) = O$ by Lemma 3.3. We have $3 - 2a_1 = \bar{H} \cdot \bar{D} \geq 0$, where \bar{H} is a general curve in $|-K_{\bar{S}} - E_1|$. Thus, it follows from the equivalence

$$K_{\bar{S}} + \lambda \bar{D} + \lambda a_1 E_1 \equiv \alpha^*(K_S + \lambda D)$$

that there is a point $Q \in E_1$ such that $(\bar{S}, \lambda \bar{D} + \lambda a_1 E_1)$ is not log canonical at the point Q .

Suppose that $Q \notin \bigcup_{i=1}^6 \bar{L}_i$. Let $\pi: \bar{S} \rightarrow \mathbb{P}^2$ be a contraction of the curves $\bar{L}_1, \dots, \bar{L}_6$. Then

$$\pi(\bar{D} + a_1 E_1) \equiv \pi(-K_{\bar{S}}) \equiv -K_{\mathbb{P}^2},$$

and π is an isomorphism in a neighborhood of Q . Let L be a general line on \mathbb{P}^2 . Then the locus

$$\text{LCS}(\mathbb{P}^2, L + \pi(\lambda \bar{D} + \lambda a_1 E_1))$$

is not connected, which is impossible by Remark 2.2.

Therefore, we may assume that $Q \in \bar{L}_1$. Put $D = aL_1 + \Upsilon$, where a is a non-negative rational number, and Υ is an effective \mathbb{Q} -divisor, whose support does not contain the line L_1 . Then

$$\tilde{\Upsilon} \equiv \alpha^*(\Upsilon) - \epsilon E_1,$$

where $\epsilon = a_1 - a/2$, and $\tilde{\Upsilon}$ is the proper transforms of the divisor Υ on the surface \bar{S} .

The log pair $(\bar{S}, \lambda a \bar{L}_1 + \lambda \tilde{\Upsilon} + \lambda(a/2 + \epsilon)E_1)$ is not log canonical at Q . Then

$$1 + a/2 - \epsilon = \bar{L}_1 \cdot \tilde{\Upsilon} > 1/\lambda - a/2 - \epsilon$$

by Remark 2.5, because $\lambda a \leq 1$. Hence, we have $a > 1/2$.

We may assume that $\text{Supp}(D)$ does not contain the conic C_1 due to Remark 2.1. Then

$$2 - 3a/2 - \epsilon = \bar{C}_1 \cdot \tilde{\Upsilon} \geq \text{mult}_Q(\tilde{\Upsilon}) > 1/\lambda - a/2 - \epsilon,$$

where \bar{C}_1 be the proper transforms of C_1 on the surface \bar{S} . Hence, we see that $a < 1/2$. \square

Lemma 3.5. *Suppose that $\Sigma = \{\mathbb{A}_1, \dots, \mathbb{A}_1\}$ and $|\Sigma| \geq 2$. Then $\text{lct}(S) = 1/2$.*

Proof. Let P be a point in Σ such that $P \neq O$. We may assume that $P \in L_1$. Then

$$2L_1 + L' \sim -K_S$$

for some line $L' \subset S$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there is an irreducible curve Z on the surface S such that

$$D = \mu Z + \Omega,$$

where μ is a rational number such that $\mu \geq 1/\lambda$, and Ω is an effective \mathbb{Q} -divisor, whose support does not contain the curve Z . Then Z is a line (see the proof of Lemma 3.4). We have

$$2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geq \mu C \cdot Z \geq \mu \geq 1/\lambda > 2,$$

where C is a general conic on S that intersects Z in two points.

We see that $\text{LCS}(S, \lambda D) = O$ and $a_1 > 1$ (see Lemma 3.3 and Remarks 2.2 and 2.7).

Arguing as in the proof of Lemma 3.4, we see that there is a point $Q \in E$ such that the singularities of the log pair $(\bar{S}, \lambda \bar{D} + \lambda a_1 E_1)$ are not log canonical at the point Q .

Suppose that $Q \in \bar{L}_1$. Let a be a non-negative rational number such that

$$D = aL_1 + \Upsilon,$$

where Υ is an effective \mathbb{Q} -divisor, whose support does not contain the line L_1 . Then

$$\tilde{\Upsilon} \equiv \alpha^*(\Upsilon) - \epsilon E_1,$$

where $\tilde{\Upsilon}$ is the proper transforms of Υ on the surface \bar{S} , and $\epsilon = a_1 - a/2$. The log pair

$$(\bar{S}, \lambda a \bar{L}_1 + \lambda \tilde{\Upsilon} + \lambda(a/2 + \epsilon)E_1)$$

is not log canonical at the point Q . We have $\bar{L}_1^2 = -1/2$. Then

$$1 - \epsilon = \bar{L}_1 \cdot \tilde{\Upsilon} > 1/\lambda - a/2 - \epsilon$$

by Remark 2.5. We have $a > 1/\lambda$, which is impossible. Hence, we see that $Q \notin \bar{L}_1$.

There is a unique reduced conic $Z \subset S$ such that $O \in Z \ni P$ and $Q \in \bar{Z}$, where \bar{Z} is the proper transform of the conic Z on the surface \bar{S} . Then $L_1 \not\subset \text{Supp}(Z)$, because $Q \notin \bar{L}_1$.

Suppose that Z is irreducible. Put $D = eZ + \Delta$, where e is a non-negative rational number, and Δ is an effective \mathbb{Q} -divisor, whose support does not contain the conic C . Then

$$\bar{\Delta} \equiv \alpha^*(\Delta) - \delta E_1,$$

where $\bar{\Delta}$ is the proper transforms of Δ on the surface \bar{S} , and $\delta = a_1 - e/2$. Then

$$2 - e - \delta = \bar{Z} \cdot \bar{\Delta} > 1/\lambda - e/2 - \delta > 2 - e/2 - \delta$$

by Remark 2.5, because $\bar{Z}^2 = 1/2$. We have $e < 0$, which is impossible.

We see that the conic Z is reducible. Then

$$Z = L_2 + L'_2,$$

where L'_2 is a line on S such that $P \in L'_2$ and $L_2 \cap L'_2 \neq \emptyset$.

The intersection $L_2 \cap L'_2$ consists of a single point. The impossibility of the case $Q \in \bar{L}_1$ implies that the surface S is smooth at the point $L_2 \cap L'_2$. There is a rational number $c \geq 0$ such that

$$D = cL_2 + \Xi,$$

where Ξ is an effective \mathbb{Q} -divisor, whose support does not contain the line L_2 . Then

$$\bar{\Xi} \equiv \alpha^*(\Xi) - v E_1,$$

where $\bar{\Xi}$ is the proper transforms of Ξ on the surface \bar{S} , and $v = a_1 - c/2$. The log pair

$$(\bar{S}, \lambda c \bar{L}_2 + \lambda \bar{\Xi} + \lambda(c/2 + v)E_1)$$

is not log canonical at Q . We have $Q \in \bar{L}_2$ and $\bar{L}_2^2 = -1$. Then

$$1 + c/2 - v = \bar{L}_2 \cdot \bar{\Xi} > 1/\lambda - c/2 - v > 2 - c/2 - v$$

by Remark 2.5. Therefore, the inequality $c > 1$ holds.

There is a unique hyperplane section T of the surface S such that $T = C_2 + L_2$ and

$$Q = \bar{C}_2 \cap \bar{L}_2 = O,$$

where C_2 is a conic, and \bar{C}_2 is the proper transforms of C_2 on the surface \bar{S} .

The conic C_2 is irreducible. We may assume that $C_2 \not\subset \text{Supp}(D)$ (see Remark 2.1). Then

$$2 - 3c/2 - v = \bar{C}_2 \cdot \bar{\Xi} \geq \text{mult}_Q(\bar{\Xi}) > 1/\lambda - c/2 - v,$$

which implies that $c < 0$. The obtained contradiction completes the proof. \square

Lemma 3.6. *Suppose that $\Sigma = \{\mathbb{D}_4\}$. Then $\text{lct}(S) = 1/3$.*

Proof. We have $r = 3$, and L_1, L_2, L_3 lie in a single plane. Then

$$\left(S, \frac{1}{3}(L_1 + L_2 + L_3)\right)$$

is log canonical and not log terminal. We may assume that $L_3 \not\subseteq \text{Supp}(D)$ due to Remark 2.1.

Let $\beta: \check{S} \rightarrow S$ be a birational morphism such that the morphism α contracts one irreducible rational curve E that contains three singular points O_1, O_2, O_3 of type \mathbb{A}_1 .

Let \check{D} and \check{L}_i be proper transforms of D and L_i on the surface \check{S} , respectively. Then

$$\check{L}_i \equiv \beta^*(L_i) - E, \quad \check{D} \equiv \beta^*(D) - \mu E,$$

where μ is a rational number. We have

$$0 \leq \check{D} \cdot \check{L}_3 = (\beta^*(D) - \mu E) \cdot \check{L}_3 = 1 - \mu E \cdot \check{L}_3 = 1 - \mu/2,$$

which implies that $\mu \leq 2$. Therefore, we may assume there is a point $Q \in E$ such that the singularities of the log pair $(\check{S}, \lambda\check{D} + \check{\mu}E)$ are not log canonical at the point Q (see Lemma 3.2).

Suppose that \check{S} is smooth at Q . The log pair $(\check{S}, \lambda\check{D} + E)$ is not log canonical at Q . Then

$$1 \geq \mu/2 = -\mu E^2 = E \cdot \check{D} > 1/\lambda > 3$$

by Remark 2.5. We see that $Q = O_j$ for some j .

The curves \check{L}_1, \check{L}_2 and \check{L}_3 are disjoint, and each of them passes through a singular point of the surface \check{S} . Therefore, we may assume that $O_i \in \check{L}_i$ for every i .

Let $\gamma: \hat{S} \rightarrow \check{S}$ be a blow up of the point O_j , and G be the exceptional curve of γ . Then

$$\hat{L}_j \equiv \gamma^*(\check{L}_j) - \frac{1}{2}G \equiv (\beta \circ \gamma)^*(L_j) - \hat{E} - G,$$

where \hat{L}_j and \hat{E} are proper transforms of the curves \check{L}_j and E on the surface \hat{S} , respectively.

Let \hat{D} be the proper transform of the divisor \check{D} on the surface \hat{S} . Then

$$\hat{E} \equiv \gamma^*(E) - \frac{1}{2}G, \quad \hat{D} \equiv \gamma^*(\check{D}) - \epsilon G \equiv (\beta \circ \gamma)^*(D) - \mu\hat{E} - (\mu/2 + \epsilon)G,$$

where ϵ is a rational number. Then $\lambda\epsilon + \lambda\mu/2 > 1/2$ (see Remark 2.7).

Suppose that $j = 3$. Then

$$0 \leq \hat{D} \cdot \hat{L}_3 = ((\beta \circ \gamma)^*(D) - \mu\hat{E} - (\mu/2 + \epsilon)G) \cdot \hat{L}_3 = 1 - \mu/2 - \epsilon$$

which implies that $\epsilon + \mu/2 < 1$. But we know that $\epsilon + \mu/2 > 3/2$.

We may assume that $Q = O_1$, and the support of the divisor D contains the line L_1 . Put

$$D = aL_1 + \Omega \equiv -K_S,$$

where $a \in \mathbb{Q}$ and $a \geq 0$, and Ω is an effective \mathbb{Q} -divisor such that $L_1 \not\subseteq \text{Supp}(\Omega)$. Then

$$\hat{\Omega} \equiv (\beta \circ \gamma)^*(\Omega) - m\hat{E} - (m/2 + b)G,$$

where $\hat{\Omega}$ is the proper transform of Ω , and m and b are non-negative rational numbers. Then

$$(\beta \circ \gamma)^*(D) - \mu\hat{E} - (\mu/2 + \epsilon)G \equiv \hat{D} = a\hat{L}_1 + \hat{\Omega} \equiv (\beta \circ \gamma)^*(aL_1 + \Omega) - (a+m)\hat{E} - (a+m/2+b)G,$$

which implies that $\mu = a + m \leq 2$ and $\epsilon = a/2 + b$. We have

$$\hat{L}_1^2 = -1, \quad \hat{E}^2 = -1, \quad G^2 = -2, \quad \hat{L} \cdot \hat{E} = 0, \quad \hat{L} \cdot G = \hat{E} \cdot G = 1$$

on the surface \hat{S} . The surface \hat{S} is smooth along the curve G . Then

$$-a \leq -a + \hat{\Omega} \cdot \hat{L}_1 = (a\hat{L}_1 + \hat{\Omega}) \cdot \hat{L}_1 = ((\beta \circ \gamma)^*(-K_S) - (a+m)\hat{E} - (a+m/2+b)G) \cdot \hat{L}_1 = 1 - a - m/2 - b,$$

which implies that $m/2 + b \leq 1$ and $a + m/2 + b \leq 1 + a \leq 3$. Thus, the equivalence

$$K_{\hat{S}} + \lambda a \hat{L}_1 + \lambda \hat{\Omega} + \lambda(a+m)\hat{E} + \lambda(a+m/2+b)G \equiv (\beta \circ \gamma)^*(K_S + \lambda a L_1 + \lambda \Omega)$$

implies the existence of a point $A \in G$ such that the log pair

$$\left(\dot{S}, \lambda a \dot{L}_1 + \lambda \dot{\Omega} + \lambda(a+m)\dot{E} + \lambda(a+m/2+b)G \right)$$

is not log canonical at the point A .

Suppose that $A \notin \dot{L}_1 \cup \dot{E}$. Then $(\dot{S}, \lambda \dot{\Omega} + \lambda(a+m/2+b)G)$ is not log canonical at A , and

$$2b+a = (a\dot{L}_1 + \dot{\Omega}) \cdot G = a + \dot{\Omega} \cdot G > a+3,$$

by Remark 2.5. We see that $b > 3/2$. But $m/2+b \leq 1$. We see that $A \in \dot{L}_1 \cup \dot{E}$.

Suppose that $A \notin \dot{L}_1$. The log pair $(\dot{S}, \lambda \dot{\Omega} + \lambda(a+m)\dot{E} + \lambda(a+m/2+b)G)$ is not log canonical at the point A . Arguing as in the previous case, we see that

$$m/2-b = (a\dot{L}_1 + \dot{\Omega}) \cdot \dot{E} = \dot{\Omega} \cdot \dot{E} \geq \text{mult}_A(\dot{\Omega}|_{\dot{E}}) > 3 - (a+m/2+b),$$

which implies that $a+m > 3$. But $a+m \leq 2$. We see that $A \in \dot{L}_1$.

The log pair $(\dot{S}, \lambda a \dot{L}_1 + \lambda \dot{\Omega} + \lambda(a+m/2+b)G)$ is not log canonical at the point A . Then

$$1-a-m/2-b = (a\dot{L}_1 + \dot{\Omega}) \cdot \dot{L}_1 = -a + \dot{\Omega} \cdot \dot{L}_1 > -a+3-(a+m/2+b)$$

by Remark 2.5. We have $a > 2$. But $a+m \leq 2$. □

Lemma 3.7. *Suppose that $\Sigma = \{\mathbb{D}_5\}$. Then $\text{lct}(S) = 1/4$.*

Proof. We have $r = 2$. We may assume that $-K_S \sim 2L_1 + L_2$. Then the log pair

$$\left(S, \frac{1}{4}(2L_1 + L_2) \right),$$

is not log terminal. We may assume $(S, \lambda D)$ is not log canonical at O (see Lemma 3.2).

It follows from [1] that S contains a line L such that $O \notin L$. Projecting from L , we see that there is a conic $C \subset S$ such that $O \notin C$, $-K_S \sim C + L$, and $C \cdot L = 2$. Put $P = C \cap L$. Then

$$P \cup O \subseteq \text{LCS} \left(S, \frac{3}{4}(C + L) + \lambda D \right) \subseteq P \cup O \cup C \cup L,$$

which is impossible by Remark 2.2. □

Lemma 3.8. *Suppose that $\Sigma = \{\mathbb{E}_6\}$. Then $\text{lct}(S) = 1/6$.*

Proof. We have $r = 1$. The log pair $(S, \frac{1}{2}L_1)$ is not log terminal. The surface S contains a plane cuspidal curve C such that $O \notin C$. The proof of Lemma 3.6 implies that $\text{lct}(S) = 1/6$. □

Lemma 3.9. *Suppose that $\Sigma = \{\mathbb{A}_2\}$. Then $\text{lct}(S) = 1/2$.*

Proof. We may assume that $-K_S \sim L_1 + L_2 + L_3 \sim L_4 + L_5 + L_6$. The log pair

$$\left(S, \frac{1}{2}(L_1 + L_2 + L_3) \right)$$

is log canonical and not log terminal. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

The proof of Lemma 3.4 implies that $\text{LCS}(S, \lambda D) = O$.

Let \bar{H} be a proper transform on \bar{S} of a general hyperplane section that contains O . Then

$$0 \leq \bar{H} \cdot \bar{D} = 3 - a_1 - a_2, \quad 2a_1 - a_2 = E_1 \cdot \bar{D} \geq 0, \quad 2a_2 - a_1 = E_2 \cdot \bar{D} \geq 0,$$

which implies that $a_1 \leq 2$ and $a_2 \leq 2$. There is a point $Q \in E_1 \cup E_2$ such that the singularities of the log pair $(\bar{S}, \lambda(\bar{D} + a_1 E_1 + a_2 E_2))$ are not log canonical at Q . We may assume that $Q \in E_1$, and

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_6 \cdot E_2 = 1,$$

which implies that $\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = \bar{L}_6 \cdot E_1 = 0$.

It follows from Remark 2.1 that we may assume that $\bar{L}_1 \not\subseteq \text{Supp}(D) \not\supseteq \bar{L}_4$. Then

$$1 - a_1 = \bar{D} \cdot \bar{L}_1 \geq 0, \quad 1 - a_2 = \bar{D} \cdot \bar{L}_4 \geq 0,$$

which implies that $a_1 \leq 1$ and $a_2 \leq 1$.

Suppose that $Q \notin E_2$. Then $(\bar{S}, \lambda\bar{D} + E_1)$ is not log canonical at Q . We have

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda > 2,$$

by Remark 2.5. Then $a_1 \geq 4/3$, which is impossible. Hence, we see that $Q \in E_2$.

The log pairs $(\bar{S}, \lambda\bar{D} + E_1 + a_2E_2)$ and $(\bar{S}, \lambda\bar{D} + a_1E_1 + E_2)$ are not log canonical at Q . Then

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \quad 2a_2 - a_1 = \bar{D} \cdot E_2 > 1/\lambda - a_1 > 2 - a_1$$

by Remark 2.5. Then $a_1 > 1$ and $a_2 > 1$, which is impossible. \square

Lemma 3.10. *Suppose that $\Sigma = \{\mathbb{A}_3\}$. Then $\text{lct}(S) = 1/2$.*

Proof. We have $r = 5$. Then $\bar{L}_i^2 = -1$ and $\bar{L}_i \cdot \bar{L}_j = 0$ for $i \neq j$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_3 = \bar{L}_5 \cdot E_3 = 1,$$

which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = 0$ and

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0.$$

We have $-K_S \sim L_1 + L_2 + L_3$. But it follows from elementary calculations that

$$\bar{L}_1 \equiv \alpha^*(L_1) - \frac{3}{4}E_1 - \frac{1}{2}E_2 - \frac{1}{4}E_3, \quad \bar{L}_2 \equiv \alpha^*(L_2) - \frac{3}{4}E_1 - \frac{1}{2}E_2 - \frac{1}{4}E_3, \quad \bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{2}E_1 - E_2 - \frac{1}{2}E_3,$$

which implies that $\text{lct}(S) \leq 1/2$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Suppose that there are a line $L \subset S$ and a rational number $\mu \geq 1/\lambda$ such that $D = \mu L + \Omega$, where Ω is an effective \mathbb{Q} -divisor, whose support does not contain the line L . Then

$$2 = C \cdot D = \mu C \cdot L + C \cdot \Omega \geq \mu C \cdot L > 2C \cdot L,$$

where C is a general conic on the surface S such that the divisor $C + L$ is a hyperplane section of the surface S . Then $|L \cap C| = 1$, which implies that $L = L_3$. But $L_3 \cdot C = 1$.

It follows from Remark 2.2 and Lemma 3.3 that $\text{LCS}(S, \lambda D) = O$.

Let \bar{H} be a general curve in $|-K_{\bar{S}} - \sum_{i=1}^3 E_i|$. Then

$$a_1 + a_3 \leq 3, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2,$$

because $\bar{H} \cdot \bar{D} \geq 0$, $E_1 \cdot \bar{D} \geq 0$, $E_2 \cdot \bar{D} \geq 0$, $E_3 \cdot \bar{D} \geq 0$, respectively.

We may assume that either $L_1 \not\subset \text{Supp}(D)$ or $L_3 \not\subset \text{Supp}(D)$. But

$$\bar{L}_1 \cdot \bar{D} = 1 - a_1, \quad \bar{L}_3 \cdot \bar{D} = 1 - a_2,$$

which implies that either $a_1 \leq 1$ or $a_2 \leq 1$. Similarly, we assume that either $a_3 \leq 1$ or $a_2 \leq 1$.

We have $a_1 \leq 2$, $a_2 \leq 2$, $a_3 \leq 2$. Then there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the log pair $(\bar{S}, \lambda(\bar{D} + a_1E_1 + a_2E_2 + a_3E_3))$ is not log canonical at Q . We may assume that $Q \notin E_3$.

Suppose that $Q \notin E_2$. Then $(\bar{S}, \lambda\bar{D} + E_1)$ is not log canonical at the point Q . We have

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda > 2,$$

by Remark 2.5. Then $a_1 > 3/2$ and $a_2 > 1$. But either $a_1 \leq 1$ or $a_2 \leq 1$. Contradiction.

Suppose that $Q \in E_2 \cap E_1$. Arguing as in the proof of Lemma 3.9, we see that

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \quad 2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 > 1/\lambda - a_1 > 2 - a_1$$

by Remark 2.5. Then $a_1 > 1$ and $2a_2 > 2 + a_3$, which is impossible.

We see that $Q \in E_2$ and $Q \notin E_1$. Then $(\bar{S}, \lambda\bar{D} + E_2)$ is not log canonical at Q . We have

$$2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 > 1/\lambda > 2,$$

which implies that $a_1 > 3/2$ and $a_2 > 2$. The latter is impossible. \square

Lemma 3.11. *Suppose that $\Sigma = \{\mathbb{A}_4\}$. Then $\text{lct}(S) = 1/3$.*

Proof. We have $r = 4$. Then $\bar{L}_i^2 = -1$ and $\bar{L}_i \cdot \bar{L}_j = 0$ for $i \neq j$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_3 = \bar{L}_4 \cdot E_4 = 1,$$

which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = 0$ and

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_4 \cdot E_1 = \bar{L}_4 \cdot E_2 = \bar{L}_4 \cdot E_3 = 0.$$

The equivalence $-K_S \sim 2L_3 + L_4$ holds. Similarly, we have

$$\bar{L}_3 \equiv \alpha^*(L_3) - \frac{2}{5}E_1 - \frac{4}{5}E_2 - \frac{6}{5}E_3 - \frac{3}{5}E_4, \quad \bar{L}_4 \equiv \alpha^*(L_4) - \frac{1}{5}E_1 - \frac{2}{5}E_2 - \frac{3}{5}E_3 - \frac{4}{5}E_4,$$

which implies that $\text{lct}(S) \leq 1/3$. Thus, we may assume that $(S, \lambda D)$ is not log canonical, which implies that $\text{LCS}(S, \lambda D) = O$ by Lemma 3.2. Let \bar{H} be a general curve in $|-K_{\bar{S}} - \sum_{i=1}^4 E_i|$. Then

$$3 \geq a_1 + a_4, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3,$$

because $\bar{H} \cdot \bar{D} \geq 0, E_1 \cdot \bar{D} \geq 0, E_2 \cdot \bar{D} \geq 0, E_3 \cdot \bar{D} \geq 0, E_4 \cdot \bar{D} \geq 0$, respectively.

We have $-K_S \sim L_1 + L_2 + L_3$ and $-K_S \sim 2L_3 + L_4$. But the log pairs

$$\left(S, \frac{1}{2}(L_1 + L_2 + L_3)\right) \quad \text{and} \quad \left(S, \frac{1}{3}(L_4 + 2L_3)\right)$$

are log canonical. So, we may assume that either $L_3 \not\subseteq \text{Supp}(D)$ or $L_1 \not\subseteq \text{Supp}(D) \not\subseteq L_4$. But

$$\bar{L}_3 \cdot \bar{D} = 1 - a_3, \quad \bar{L}_1 \cdot \bar{D} = 1 - a_1, \quad \bar{L}_4 \cdot \bar{D} = 1 - a_4,$$

which implies that there is a point $Q \in \cup_{i=1}^4 E_i$ such that $(\bar{S}, \lambda(\bar{D} + \sum_{i=1}^4 a_i E_i))$ is not log canonical at the point Q . Arguing as in the proof of Lemma 3.10, we see that

$$\left\{ \begin{array}{l} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 3, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 3 \text{ and } 2a_2 > 3 + a_3, \\ Q \in E_2 \setminus ((E_1 \cap E_2) \cup (E_2 \cap E_3)) \Rightarrow 2a_2 > a_1 + a_3 + 3, \\ Q \in E_2 \cap E_3 \Rightarrow 2a_2 > 3 + a_1 \text{ and } 2a_3 > 3 + a_4, \\ Q \in E_3 \setminus ((E_2 \cap E_3) \cup (E_3 \cap E_4)) \Rightarrow 2a_3 > 3 + a_2 + a_4, \\ Q \in E_3 \cap E_4 \Rightarrow 2a_3 > 3 + a_2 \text{ and } 2a_4 > 3, \\ Q \in E_4 \setminus (E_4 \cap E_3) \Rightarrow 2a_4 > 3, \end{array} \right.$$

which leads to a contradiction, because either $a_3 \leq 1$ or $a_1 \leq 1$ and $a_4 \leq 1$. \square

Lemma 3.12. *Suppose that $\Sigma = \mathbb{A}_5$. Then $\text{lct}(S) = 1/4$.*

Proof. We have $r = 3$. We may assume that $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_4 = 1$. Then

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_1 \cdot E_5 = \bar{L}_2 \cdot E_3 = 0$$

and $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_3 \cdot E_5 = 0$. But $-K_S \sim 3L_3$. Then $\text{lct}(S) \leq 1/4$, because

$$\bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{2}{3}E_5.$$

We may assume that $(S, \lambda D)$ is not log canonical. Then $\text{LCS}(S, \lambda D) = O$. by Lemma 3.2.

Let \bar{H} be a proper transform on \bar{S} of a general hyperplane section that contains O . Then

$$(3.13) \quad 3 \geq a_1 + a_5, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3 + a_5, \quad 2a_5 \geq a_4,$$

because $\bar{H} \cdot \bar{D} \geq 0, E_1 \cdot \bar{D} \geq 0, E_2 \cdot \bar{D} \geq 0, E_3 \cdot \bar{D} \geq 0, E_4 \cdot \bar{D} \geq 0, E_5 \cdot \bar{D} \geq 0$, respectively.

We may assume that $L_3 \not\subseteq \text{Supp}(D)$ due to Remark 2.1. Then $1 - a_4 = \bar{L}_3 \cdot \bar{D} \geq 0$, which easily implies that $a_1 \leq 5/2, a_2 \leq 2, a_3 \leq 3/2, a_4 \leq 1, a_5 \leq 5/4$.

There is a point $Q \in \cup_{i=1}^5 E_i$ such that the log pair $(\bar{S}, \lambda(\bar{D} + \sum_{i=1}^5 a_i E_i))$ is not log canonical at the point Q . Arguing as in the proof of Lemma 3.10, we see that

$$(3.14) \quad \left\{ \begin{array}{l} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 4, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 4 \text{ and } 2a_2 > 4 + a_3, \\ Q \in E_2 \setminus ((E_1 \cap E_2) \cup (E_2 \cap E_3)) \Rightarrow 2a_2 > a_1 + a_3 + 4, \\ Q \in E_2 \cap E_3 \Rightarrow 2a_2 > 4 + a_1 \text{ and } 2a_3 > 4 + a_4, \\ Q \in E_3 \setminus ((E_2 \cap E_3) \cup (E_3 \cap E_4)) \Rightarrow 2a_3 > 4 + a_2 + a_4, \\ Q \in E_3 \cap E_4 \Rightarrow 2a_3 > 4 + a_2 \text{ and } 2a_4 > 4 + a_5, \\ Q \in E_4 \setminus ((E_3 \cap E_4) \cup (E_4 \cap E_5)) \Rightarrow 2a_4 > 4 + a_3 + a_5, \\ Q \in E_4 \cap E_5 \Rightarrow 2a_4 > 4 + a_3 \text{ and } 2a_5 > 4, \\ Q \in E_5 \setminus (E_4 \cap E_5) \Rightarrow 2a_5 > a_4 + 4. \end{array} \right.$$

Now taking into account the inequalities 3.13, the inequalities 3.14, the inequality $a_4 \leq 4$, and the inequality $a_1 + a_5 \leq 3$, we see that either $Q = E_3 \cap E_4$ or $Q = E_4 \cap E_5$.

Let H_1 and H_3 be general curves in $|-K_S|$ that contain L_1 and L_3 , respectively. Then

$$H_1 = L_1 + C_1, \quad H_3 = L_3 + C_3,$$

where C_1 and C_3 are irreducible conics such that $C_1 \not\subseteq \text{Supp}(D) \not\supseteq C_3$.

Let \bar{C}_1 and \bar{C}_3 be the proper transforms of C_1 and C_3 on the surface \bar{S} , respectively. Then

$$\bar{C}_1 \cdot E_1 = \bar{C}_1 \cdot E_2 = \bar{C}_1 \cdot E_3 = \bar{C}_1 \cdot E_4 = \bar{C}_3 \cdot E_1 = \bar{C}_3 \cdot E_3 = \bar{C}_3 \cdot E_4 = \bar{C}_3 \cdot E_5 = 0$$

and $\bar{C}_1 \cdot E_5 = \bar{C}_3 \cdot E_2 = 1$. Therefore, we see that

$$0 \leq \bar{C}_1 \cdot \bar{D} = 2 - a_5, \quad 2 - a_2 = \bar{C}_3 \cdot \bar{D} \geq 0,$$

which implies that $a_2 \leq 2$ and $a_5 \leq 2$. Now we can easily obtain a contradiction. \square

Lemma 3.15. *Suppose that $\Sigma = \{A_1, A_5\}$. Then $\text{lct}(S) = 1/4$.*

Proof. Let P be a point in Σ of type A_1 . Then $r = 2$. We may assume that $P \in L_1$. Then

$$\bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = \bar{L}_2 \cdot E_5 = \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_1 \cdot E_5 = 0,$$

and $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_4 = 1$. The equivalence $-K_S \sim 3L_2$ holds. Then $\text{lct}(S) \leq 1/4$, because

$$\bar{L}_2 \equiv \alpha^*(L_2) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{2}{3}E_5.$$

We may assume that $(S, \lambda D)$ is not log canonical. Then $\text{LCS}(S, \lambda D) \subseteq \{O, P\}$ by Lemma 3.2.

Suppose that $(S, \lambda D)$ is not log terminal at P . Let $\beta: \check{S} \rightarrow S$ be a blow up of P . Then

$$\check{D} \equiv \beta^*(-K_S) - mF,$$

where F is the β -exceptional curve, \check{D} is the proper transform of the divisor D , and $m \in \mathbb{Q}$. Then

$$0 \leq \check{H} \cdot \check{D} = \left(\beta^*(-K_S) - mF\right) \cdot \left(\beta^*(-K_S) - F\right) = 3 - 2m,$$

where \check{H} is general curve in $|\beta^*(-K_S) - F|$. Thus, we have $m \leq 3/2$. But $m > 2$ by Remark 2.7.

We see that $\text{LCS}(S, \lambda D) = O$. Let C_1 and C_2 be general conics on the surface S such that

$$L_1 + C_1 \sim L_2 + C_2 \sim -K_S,$$

and let \bar{C}_1 and \bar{C}_2 be the proper transforms of C_1 and C_2 on the surface \bar{S} , respectively. Then

$$2 - a_1 = \bar{C}_1 \cdot \bar{D} \geq 0, \quad 2 - a_5 = \bar{C}_2 \cdot \bar{D} \geq 0,$$

because $C_1 \not\subseteq \text{Supp}(D) \not\supseteq C_2$. We may assume that $L_2 \not\subseteq \text{Supp}(D)$ due to Remark 2.1.

Arguing as in the proof of Lemma 3.12, we obtain the inequalities

$$3 \geq a_1 + a_5, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1 + a_3, \quad 2a_3 \geq a_2 + a_4, \quad 2a_4 \geq a_3 + a_5, \quad 2a_5 \geq a_4, \quad 2 \geq a_2, \quad 2 \geq a_5, \quad 1 \geq a_4,$$

which imply that there is a point $Q \in \cup_{i=1}^5 E_i$ such that $(\bar{S}, \lambda(\bar{D} + \sum_{i=1}^5 a_i E_i))$ is not log canonical at the point Q . Arguing as in the proof of Lemma 3.10, we obtain a contradiction. \square

Lemma 3.16. *Suppose that $\Sigma = \{A_1, A_4\}$. Then $\text{lct}(S) = 1/3$.*

Proof. We have $r = 3$. Let P be a point in Σ of type A_1 . We may assume that $P \in L_1$. Then

$$\bar{L}_1 \cdot E_1 = 1, \quad \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = 0,$$

and we may assume that $\bar{L}_3 \cdot E_3 = \bar{L}_2 \cdot E_4 = 1$. Then

$$\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = 0.$$

The equivalence $-K_S \sim L_2 + 2L_3$ holds. But

$$\bar{L}_2 \equiv \alpha^*(L_2) - \frac{1}{5}E_1 - \frac{2}{5}E_2 - \frac{3}{5}E_3 - \frac{4}{5}E_4, \quad \bar{L}_3 \equiv \alpha^*(L_3) - \frac{2}{5}E_1 - \frac{4}{5}E_2 - \frac{6}{5}E_3 - \frac{3}{5}E_4,$$

which implies that $\text{lct}(S) \leq 1/3$. Thus, we may assume that $(S, \lambda D)$ is not log canonical.

We may assume that either $L_3 \not\subseteq \text{Supp}(D)$ or $L_1 \not\subseteq \text{Supp}(D) \not\supseteq L_2$ (see Remark 2.1).

Arguing as in the proof of Lemma 3.15, we see that the log pair $(S, \lambda D)$ is log canonical outside of the point O . Now arguing as in the proof of Lemma 3.11, we obtain a contradiction. \square

Lemma 3.17. *Suppose that $\Sigma = \{A_1, A_3\}$. Then $\text{lct}(S) = 1/2$.*

Proof. Let P be a point in Σ of type A_1 . We may assume that $P \in L_1$. Then $r = 4$, and it easily follows from [1] that the surface S contains lines L_5, L_6, L_7 such that

$L_5 \ni P \in L_6$, $O \notin L_7 \not\ni P$, $L_3 \cap L_5 \neq \emptyset$, $L_4 \cap L_6 \neq \emptyset$, $L_7 \cap L_2 \neq \emptyset$, $L_7 \cap L_5 \neq \emptyset$, $L_7 \cap L_6 \neq \emptyset$, which implies that $L_7 \cap L_1 = L_7 \cap L_3 = L_7 \cap L_4 = \emptyset$. Then $-K_S \sim L_1 + L_3 + L_5$ and

$$L_1 + L_3 + L_5 \sim L_1 + L_4 + L_6 \sim L_5 + L_6 + L_7 \sim L_2 + 2L_1 \sim L_2 + L_3 + L_4 \sim 2L_2 + L_7,$$

which implies that $\text{lct}(S) \leq 1/2$. Hence, we may assume that $(S, \lambda D)$ is not log canonical.

Put $D = \mu_i L_i + \Omega_i$, where μ_i is a non-negative rational number, and Ω_i is an effective \mathbb{Q} -divisor, whose support does not contain the line L_i . Let us show that $\mu_i < 1/\lambda$ for $i = 1, \dots, 7$.

Suppose that $\mu_2 \geq 1/\lambda$. We may assume that $L_1 \not\subseteq \text{Supp}(D)$ by Remark 2.1. Then

$$1 = L_1 \cdot D = L_1 \cdot (\mu_2 L_2 + \Omega_2) \geq \mu_2 L_1 \cdot L_2 = \mu_2/2 > 1,$$

which is a contradiction. Similarly, we see that $\mu_i < 1/\lambda$ for $i = 1, \dots, 7$.

Arguing as in the proof of Lemma 3.4, we see that $\text{LCS}(S, \lambda D)$ does not contain curves and smooth points of the surface S . Then either $\text{LCS}(S, \lambda D) = O$ or $\text{LCS}(S, \lambda D) = P$ by Remark 2.2.

Suppose that $\text{LCS}(S, \lambda D) = P$. Put

$$D = \mu_5 L_5 + \mu_6 L_6 + \Upsilon,$$

where Υ is an effective \mathbb{Q} -divisor such that $L_5 \not\subseteq \text{Supp}(\Upsilon) \not\subseteq L_6$. Then $\mu_5 > 0$ and $\mu_6 > 0$. But

$$1 = L_7 \cdot D = L_7 \cdot (\mu_5 L_5 + \mu_6 L_6 + \Upsilon) \geq L_7 \cdot (\mu_5 L_5 + \mu_6 L_6) = \mu_5 + \mu_6,$$

because we may assume that $L_7 \not\subseteq \text{Supp}(\Upsilon)$. Let $\beta: \check{S} \rightarrow S$ be a blow up of the point P . Then

$$\mu_5 \check{L}_5 + \mu_6 \check{L}_6 + \check{\Upsilon} \equiv \beta^*(\mu_5 L_5 + \mu_6 L_6 + \Upsilon) - (\mu_5/2 + \mu_6/2 + \epsilon)G,$$

where ϵ is a rational number, G is the exceptional curve of β , and $\check{L}_5, \check{L}_6, \check{\Upsilon}$ are proper transforms of the divisors L_5, L_6, Υ on the surface \check{S} , respectively. Then

$$0 \leq (\mu_5 \check{L}_5 + \mu_6 \check{L}_6 + \check{\Upsilon})\check{H} = 3 - \mu_5 - \mu_6 - 2\epsilon,$$

where \check{H} is a general curve in $|-K_{\check{S}} - G|$. There is a point $Q \in G$ such that the singularities of the log pair $(\check{S}, \lambda(\mu_5 \check{L}_5 + \mu_6 \check{L}_6 + \check{\Upsilon}) + \lambda(\mu_5/2 + \mu_6/2 + \epsilon)G)$ are not log canonical at Q . We have

$$2 - 2\epsilon = \check{\Upsilon} \cdot (\check{L}_5 + \check{L}_6) \geq 0,$$

which implies that $\epsilon \leq 1$. Then $2\epsilon = \check{\Omega} \cdot G > 2$ in the case when $Q \notin \check{L}_5 \cup \check{L}_6$ by Remark 2.5, which implies that we may assume that $Q \in \check{L}_5$. Then

$$1 + \mu_5/2 - \mu_6 - \epsilon = \check{\Omega} \cdot \check{L}_5 > 2 - \mu_5/2 - \mu_6/2 - \epsilon,$$

due to Remark 2.5. Thus, we see that $\mu_5 > 1$. But $\mu_5 \leq \mu_5 + \mu_6 \leq 1$.

We see that $\text{LCS}(S, \lambda D) = O$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_4 \cdot E_3 = 1,$$

and $\bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_3 = \bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_4 \cdot E_2 = 0$. The log pairs

$$\left(S, \frac{1}{2}(2L_1 + L_2)\right) \text{ and } \left(S, \frac{1}{2}(L_2 + L_3 + L_4)\right)$$

are log canonical. So, we may assume that either $L_2 \not\subseteq \text{Supp}(D)$ or $L_1 \not\subseteq \text{Supp}(D) \not\subseteq L_3$, which easily leads to a contradiction (see the proof of Lemma 3.10). \square

Lemma 3.18. *Suppose that $\Sigma = \{A_1, A_2\}$. Then $\text{lct}(S) = 1/2$.*

Proof. Let P be a point in Σ of type A_1 . We may assume that $P \in L_1$. Then $r = 5$, and it easily follows from [1] that the surface S contains lines $L_6, L_7, L_8, L_9, L_{10}, L_{11}$ such that

$$P = L_1 \cap L_6 \cap L_7 \cap L_8, \quad L_9 \cap L_6 \neq \emptyset, \quad L_9 \cap L_7 \neq \emptyset, \quad L_9 \cap L_8 \neq \emptyset$$

and $L_9 \cap L_7 \neq \emptyset$, $L_{10} \cap L_7 \neq \emptyset$, $L_{10} \cap L_8 \neq \emptyset$, $L_{11} \cap L_6 \neq \emptyset$, $L_{11} \cap L_8 \neq \emptyset$. Then

$$L_2 \not\ni P \notin L_3, \quad L_4 \not\ni P \notin L_5, \quad L_6 \not\ni O \notin L_7, \quad L_8 \not\ni O \notin L_9, \quad L_{10} \not\ni O \notin L_{11},$$

which implies that $-K_S \sim L_3 + L_4 + L_5 \sim 2L_1 + L_2 \sim L_3 + L_4 + L_5$ and

$$2L_1 + L_2 \sim L_1 + L_3 + L_6 \sim L_1 + L_4 + L_7 \sim L_1 + L_5 + L_8 \sim L_6 + L_7 + L_9 \sim L_7 + L_8 + L_{10} \sim L_6 + L_8 + L_{11}.$$

We see that $\text{lct}(S) \leq 1/2$. Therefore, we may assume that $(S, \lambda D)$ is not log canonical.

Arguing as in the proof of Lemma 3.17, we see that $\text{LCS}(S, \lambda D) = O$. We may assume that $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = 1$, $\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0$.

It follows from elementary calculations that

$$\bar{L}_1 \equiv \alpha^*(L_1) - \frac{2}{3}E_1 - \frac{1}{3}E_2, \quad \bar{L}_2 \equiv \alpha^*(L_2) - \frac{2}{3}E_1 - \frac{1}{3}E_2,$$

which implies that we may assume that either $L_1 \not\subseteq \text{Supp}(D)$ or $L_2 \not\subseteq \text{Supp}(D)$. But

$$\bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{3}E_1 - \frac{2}{3}E_2, \quad \bar{L}_4 \equiv \alpha^*(L_4) - \frac{1}{3}E_1 - \frac{2}{3}E_2, \quad \bar{L}_5 \equiv \alpha^*(L_5) - \frac{1}{3}E_1 - \frac{2}{3}E_2,$$

which easily implies that we may assume that the support of the divisor D does not contain one of the lines L_3, L_4, L_5 . Arguing as in the proof of Lemma 3.9, we obtain a contradiction. \square

Lemma 3.19. *Suppose that $\Sigma = \{\mathbb{A}_2, \dots, \mathbb{A}_2\}$ and $|\Sigma| \geq 2$. Then $\text{lct}(S) = 1/3$.*

Proof. Let P be a point in Σ such that $P \neq O$. We may assume that $P \in L_1$. Then $-K_S \sim 3L_1$, which implies that $\text{lct}(S) \leq 1/3$. Thus, we may assume that $(S, \lambda D)$ is not log canonical.

We may assume that $(S, \lambda D)$ is not log canonical at the point O by Lemma 3.2. Then

$$\bar{L}_1 \equiv \alpha^*(L_1) - \frac{1}{3}E_1 - \frac{2}{3}E_2,$$

where we assume that $\bar{L}_1 \cap E_2 \neq \emptyset$. Thus, we may assume that $L_1 \not\subseteq \text{Supp}(D)$ due to Remark 2.1, which implies that $a_2 \leq 1$, because $\bar{D} \cdot \bar{L}_1 \geq 0$. Arguing as in the proof of Lemma 3.9, we see that

$$3 \geq a_1 + a_2 \leq 3, \quad 2a_1 \geq a_2, \quad 2a_2 \geq a_1, \quad 1 \geq a_2,$$

which implies that there is a point $Q \in E_1 \cup E_2$ such that the log pair $(\bar{S}, \lambda(\bar{D} + a_1E_1 + a_2E_2))$ is not log canonical at the point Q . Arguing as in the proof of Lemma 3.9, we see that

$$\begin{cases} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 3, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 3 \text{ and } 2a_2 > 3, \\ Q \in E_2 \setminus (E_2 \cap E_1) \Rightarrow 2a_2 > a_1 + 3, \end{cases}$$

which easily leads to a contradiction. \square

Lemma 3.20. *Suppose that $\Sigma = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_2\}$. Then $\text{lct}(S) = 1/3$.*

Proof. Let $P \neq O$ be a point in Σ of type \mathbb{A}_2 . We may assume that $P \in L_1$. Then $-K_S \sim 3L_1$, which implies that $\text{lct}(S) \leq 1/3$. Thus, we may assume that $(S, \lambda D)$ is not log canonical.

We may assume that $L_1 \not\subseteq \text{Supp}(D)$ due to Remark 2.1. But $\text{LCS}(S, \lambda D) \subseteq \Sigma$ by Lemma 3.2.

Arguing as in the proof of Lemma 3.15, we see that $\text{LCS}(S, \lambda D) \subseteq O \cup P$, which easily leads to a contradiction (see the proof of Lemma 3.19). \square

Lemma 3.21. *Suppose that $\Sigma = \{\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_3\}$. Then $\text{lct}(S) = 1/2$.*

Proof. Let P_1 and P_2 be points in Σ of type \mathbb{A}_1 . Then we may assume that $P_1 \in L_1$ and $P_2 \in L_2$, while we have $r = 3$. The surface S contains lines L_4 and L_5 such that

$$P_1 \in L_4 \ni P_2, \quad O \notin L_4, \quad P_1 \notin L_3 \not\ni P_2, \quad L_5 \cap \Sigma = \emptyset,$$

which implies that $L_5 \cap L_3 \neq \emptyset$, $L_5 \cap L_4 \neq \emptyset$, $L_5 \cap L_1 = \emptyset$, $L_5 \cap L_2 = \emptyset$. Then

$$(3.22) \quad -K_S \sim L_1 + L_2 + L_4 \sim L_3 + 2L_1 \sim L_3 + 2L_2 \sim 2L_3 + L_5 \sim 2L_4 + L_5,$$

which implies that $\text{lct}(S) \leq 1/2$. We may assume that $(S, \lambda D)$ is not log canonical.

Put $D = \mu_i L_i + \Omega_i$, where μ_i is a non-negative number, and Ω_i is an effective \mathbb{Q} -divisor, whose support does not contain the line L_i . Let us show that $\mu_i < 1/\lambda$ for every $i = 1, \dots, 5$.

Suppose that $\mu_1 \geq 1/\lambda > 2$. It follows from equivalences 3.22 and Remark 2.1 that we may assume that $L_3 \not\subseteq \text{Supp}(D)$. Therefore, we have

$$1 = L_3 \cdot D = L_3 \cdot (\mu_1 L_1 + \Omega_1) \geq \mu_1 L_3 \cdot L_1 = \mu_1/2 > 1,$$

which is a contradiction. Similarly, we see that $\mu_2 < 1/\lambda$, $\mu_3 < 1/\lambda$, $\mu_4 < 1/\lambda$, $\mu_5 < 1/\lambda$.

Arguing as in the proof of Lemma 3.4, we see that $\text{LCS}(S, \lambda D)$ does not contain curves and smooth points of S . It follows from Remark 2.2 that $\text{LCS}(S, \lambda D)$ consist of one point in Σ .

Suppose that $\text{LCS}(S, \lambda D) = P_1$. Let $\beta: \check{S} \rightarrow S$ be a blow up of the point P_1 . Then

$$\mu_4 \check{L}_4 + \check{\Omega} \equiv \beta^*(\mu_4 L_4 + \Omega) - (\mu_4/2 + \epsilon)G,$$

where G is the exceptional curve of the birational morphism β , \check{L}_4 and $\check{\Omega}$ are proper transforms of the divisors L_4 and Ω on the surface \check{S} , respectively, and ϵ is a positive rational number. Then

$$0 \leq (\mu_4 \check{L}_4 + \check{\Omega})\check{H} = (\beta^*(\mu_4 L_4 + \Omega) - (\mu_4/2 + \epsilon)G) \cdot (\beta^*(-K_S) - G) = 3 - \mu_4 - 2\epsilon,$$

where \check{H} is a general curve in $|-K_{\check{S}} - G|$. Thus, there is a point $P \in G$ such that the log pair

$$(\check{S}, \mu_4 \check{L}_4 + \check{\Omega} + (\mu_4/2 + \epsilon)G)$$

is not log canonical at P . We have $1 - \epsilon = \check{\Omega} \cdot \check{L}_4 \geq 0$, which implies that $\epsilon \leq 1$. Then

$$2\epsilon = \check{\Omega} \cdot G > 2$$

in the case when $P \notin \check{L}_4$ (see Remark 2.5). Thus, we see that $P \in \check{L}_4$. Then

$$1 - \epsilon = \check{\Omega} \cdot \check{L}_4 > 2 - \mu_4/2 - \epsilon,$$

due to Remark 2.5. Thus, we see that $\mu_4 > 2$, which is a contradiction.

Similarly, we see that $P_2 \notin \text{LCS}(S, \lambda D)$. Then $\text{LCS}(S, \lambda D) = O$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_3 = \bar{L}_3 \cdot E_2 = 1, \quad \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_3 = 0.$$

It follows from elementary calculations that

$$\bar{L}_1 \equiv \alpha^*(L_1) - \frac{3}{4}E_1 - \frac{1}{2}E_2 - \frac{1}{4}E_3, \quad \bar{L}_2 \equiv \alpha^*(L_2) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3, \quad \bar{L}_3 \equiv \alpha^*(L_3) - \frac{1}{2}E_1 - E_2 - \frac{1}{2}E_3,$$

which implies that the singularities of the log pairs

$$\left(S, \frac{1}{2}(2L_1 + L_3)\right) \quad \text{and} \quad \left(S, \frac{1}{2}(2L_2 + L_3)\right)$$

are log canonical. But we may assume that either $L_1 \not\subseteq \text{Supp}(D) \not\supseteq L_2$ or $L_3 \not\subseteq \text{Supp}(D)$, because the equivalences 3.22 hold. Now the proof of Lemma 3.10 leads to a contradiction. \square

Lemma 3.23. *Suppose that $\Sigma = \{\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_2\}$. Then $\text{lct}(S) = 1/2$.*

Proof. Let $P_1 \neq P_2$ be points in Σ of type \mathbb{A}_1 . Then we may assume that $P_1 \in L_1$ and $P_2 \in L_4$, while we have $r = 4$. The surface S contains lines L_5, L_6, L_7, L_8 such that

$$P_1 \in L_5, \quad P_2 \in L_6, \quad P_1 \in L_7 \ni P_2, \quad O \notin L_8, \quad P_1 \notin L_8 \not\ni P_2,$$

which implies that $L_8 \cap L_7 \neq \emptyset$, $L_8 \cap L_2 \neq \emptyset$, $L_8 \cap L_3 \neq \emptyset$, $L_2 \cap L_7 = \emptyset$, $L_3 \cap L_7 = \emptyset$. Then $L_1 + L_4 + L_7 \sim L_2 + 2L_1 \sim L_3 + 2L_4 \sim 2L_7 + L_8 \sim L_2 + L_3 + L_8 \sim L_1 + L_3 + L_5 \sim L_4 + L_2 + L_6$, and $-K_S \sim L_1 + L_4 + L_7$. Then $\text{lct}(S) \leq 1/2$. We may assume that $(S, \lambda D)$ is not log canonical.

Arguing as in the proof of Lemma 3.21, we see that $\text{LCS}(S, \lambda D) = O$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_2 = 1, \quad \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_1 = 0.$$

The log pair $(S, L_1 + \frac{1}{2}L_2)$ is log canonical, because the equivalences

$$\bar{L}_1 \equiv \alpha^*(L_1) - \frac{2}{3}E_1 - \frac{1}{3}E_2, \quad \bar{L}_2 \equiv \alpha^*(L_2) - \frac{2}{3}E_1 - \frac{1}{3}E_2$$

hold. So, we may assume that either $L_1 \not\subseteq \text{Supp}(D)$ or $L_2 \not\subseteq \text{Supp}(D)$, because $-K_S \sim 2L_1 + L_2$.

Similarly, we may assume that either $L_3 \not\subseteq \text{Supp}(D)$ or $L_4 \not\subseteq \text{Supp}(D)$, which very easily leads to a contradiction (see the proof of Lemma 3.9). \square

Therefore, it follows from [1] that the assertion of Theorem 1.8 is proved.

4. INVARIANT THRESHOLDS.

In this section we prove the following two lemmas.

Lemma 4.1. *Let S be a cubic surface in \mathbb{P}^3 given by the equation 1.9. Then $\text{lct}(S, S_4) = 1$.*

Proof. Let O_1, \dots, O_4 be singular points of the surface S , and let L_{ij} be a line in S that contains the points O_i and O_j , where $i \neq j$. Then S_4 acts transitively on $\{O_1, \dots, O_4\}$ and $\{L_{12}, \dots, L_{34}\}$.

Let T be a curve that is cut out on S by the equation $x + y + z + t = 0$. Then T is S_4 -invariant, which implies that $\text{lct}(S, S_4) \leq 1$. Suppose that $\text{lct}(S, S_4) < 1$. Then there is an effective S_4 -invariant \mathbb{Q} -divisor D such that $D \equiv -K_S$, and $(S, \lambda D)$ is not log canonical, where $\lambda \in \mathbb{Q}$ and $\lambda < 1$.

The surface S does not contain S_4 -invariant points, because the group S_4 does not have faithful two-dimensional linear representations. Then $\text{LCS}(S, \lambda D)$ contains a curve by Remark 2.2.

There are a reduced S_4 -invariant curve $C \subset S$ and a rational number $m \geq 1/\lambda$ such that

$$D = mC + \Omega,$$

where Ω is an effective divisor, whose support does not contain components of C . Then

$$3 = -K_S \cdot D = m \deg(C) - K_S \cdot \Omega \geq m \deg(C) > \deg(C),$$

which implies that either C is a line, or C is a conic.

Suppose that the curve C is not an irreducible conic. Let L be any irreducible component of the curve C . Then L is a line. Let M be a general hyperplane section of S that contains L . Then

$$M = L + \bar{L} \sim -K_S,$$

where \bar{L} is an irreducible conic. We have

$$2 = \bar{L} \cdot D = m \bar{L} \cdot L + \bar{L} \cdot \Omega \geq m \bar{L} \cdot C > \bar{L} \cdot L,$$

which implies that $L \cap \{O_1, \dots, O_4\} \neq \emptyset$. Then $L \in \{L_{12}, \dots, L_{34}\}$, which is impossible, because the curve C contains at most two components.

We see that $\text{LCS}(S, \lambda D)$ does not contain lines, and C is an irreducible conic.

Let R be a hyperplane section of the surface S that contains the conic C . Then

$$R = C + \bar{C} \sim -K_S,$$

where \bar{C} is a S_4 -invariant line. The intersection $\bar{C} \cap C$ consists of two points.

The log pair $(S, \bar{C} + C)$ is log canonical. We may assume $\bar{C} \not\subseteq \text{Supp}(\Omega)$ by Remark 2.1. Then

$$1 = \bar{S} \cdot D = m \bar{C} \cdot C + \bar{C} \cdot \Omega \geq m \bar{C} \cdot C > \bar{C} \cdot C,$$

which implies that $\bar{C} \cap C \subset \{O_1, \dots, O_4\}$. Then $\bar{C} \in \{L_{12}, \dots, L_{34}\}$, which is impossible. \square

Lemma 4.2. *Let S be a cubic surface in \mathbb{P}^3 given by the equation 1.10. Then $\text{lct}(S, S_3 \times \mathbb{Z}_3) = 1$.*

Proof. Put $G = S_3 \times \mathbb{Z}_3$. Let O_1, O_2, O_3 be singular points of the surface S , and $L_i \subset S$ be a line such that $O_i \notin L_i$. Then $\text{lct}(S, G) \leq 1$, because the curve $L_1 + L_2 + L_3$ is G -invariant.

We suppose that $\text{lct}(S, G) < 1$. Then there is an effective G -invariant \mathbb{Q} -divisor D such that the equivalence $D \equiv -K_S$ holds, and $(S, \lambda D)$ is not log canonical, where $\lambda \in \mathbb{Q}$ and $\lambda < 1$.

The surface S does not contain G -invariant points. Then $|\text{LCS}(S, \lambda D)| = +\infty$ by Remark 2.2, which implies that there are a G -invariant curve $C \subset S$ and a rational number $m \geq 1/\lambda$ such that

$$D = mC + \Omega,$$

where Ω is an effective \mathbb{Q} -divisor, whose support does not contain components of C . Then

$$3 = -K_S \cdot D = m \deg(C) - K_S \cdot \Omega \geq m \deg(C) > \deg(C),$$

which implies that either C is a line, or C is a conic.

The only lines contained in S are the lines L_1, L_2, L_3 . The group G acts on the set

$$\{L_1, L_2, L_3\}$$

transitively. Hence, the curve C is neither a line, nor conic. \square

5. FIBERWISE MAPS.

Let Z be a smooth curve. Suppose that there is a commutative diagram

$$(5.1) \quad \begin{array}{ccc} V & \xrightarrow{\rho} & \bar{V} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ Z & \xlongequal{\quad} & Z \end{array}$$

such that π and $\bar{\pi}$ are flat morphisms, and ρ is a birational map that induces an isomorphism

$$(5.2) \quad \rho|_{V \setminus X}: V \setminus X \longrightarrow \bar{V} \setminus \bar{X},$$

where X and \bar{X} are scheme fibers of π and $\bar{\pi}$ over a point $O \in Z$, respectively. Suppose that

- the varieties V and \bar{V} have terminal \mathbb{Q} -factorial singularities,
- the divisors $-K_V$ and $-K_{\bar{V}}$ are π -ample and $\bar{\pi}$ -ample, respectively,
- the fibers X and \bar{X} are irreducible.

The following example is due to [6].

Example 5.3. Suppose that X is a smooth cubic surface that contains lines L_1, L_2, L_3 such that the intersection $L_1 \cap L_2 \cap L_3$ consists of single point $P \in X$. There is commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & \bar{U} \\ \alpha \downarrow & & \downarrow \beta \\ V & \xrightarrow{\rho} & \bar{V}, \end{array}$$

where α is a blow up of P , ψ is an antiflip in the proper transforms of L_1, L_2, L_3 , and β is a contraction of the proper transform of the fiber X . Then \bar{X} is a cubic surface that has one singular point of type \mathbb{D}_4 . We have $\text{lct}(X) = 2/3$ and $\text{lct}(\bar{X}) = 1/3$ (see Example 1.7 and Lemma 3.6).

Which kind of conditions on the fibers X and \bar{X} imply that ρ is biregular?

Example 5.4. Suppose that both fibers X and \bar{X} are nonsingular del Pezzo surfaces such that the inequality $K_X^2 = K_{\bar{X}}^2 \leq 4$ holds. Then ρ is an isomorphism (see [14]).

The question we asked is local by the curve Z . Thus, in the following, we will not assume that the curve Z is projective. Let us consider two examples with $Z = \mathbb{C}^1$ (see [14]).

Example 5.5. Let V be \bar{V} subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

$$x^3 + y^2z + z^2w + t^{12}w^3 = 0 \text{ and } x^3 + y^2z + z^2w + w^3 = 0,$$

respectively, where t is a coordinate on \mathbb{C}^1 , and (x, y, z, w) are coordinates on \mathbb{P}^3 . The projections

$$\pi: V \longrightarrow \mathbb{C}^1 \text{ and } \bar{\pi}: \bar{V} \longrightarrow \mathbb{C}^1$$

are fibrations into cubic surfaces. Let O be the point on \mathbb{C}^1 given by $t = 0$. Then \bar{X} is smooth, the surface X has one singular point of type \mathbb{E}_6 . Put $Z = \mathbb{C}^1$. Then the map

$$(x, y, z, w) \longrightarrow (t^2x, t^3y, z, t^6w)$$

induces a birational map $\rho: V \dashrightarrow \bar{V}$ such that the diagrams 5.1 and isomorphism 5.2 exist, and ρ is not biregular. But $\text{lct}(X) = 1/6$ and $\text{lct}(\bar{X}) = 2/3$ (see Example 1.7 and Lemma 3.8).

Example 5.6. Let V be \bar{V} subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

$$wz^2 + zx^2 + y^2x + t^8w^3 = 0 \text{ and } wz^2 + zx^2 + y^2x + w^3 = 0,$$

respectively, where t is a coordinate on \mathbb{C}^1 , and (x, y, z, w) are coordinates on \mathbb{P}^3 . The projections

$$\pi: V \longrightarrow \mathbb{C}^1 \text{ and } \bar{\pi}: \bar{V} \longrightarrow \mathbb{C}^1$$

are fibrations into cubic surfaces. Let O be the point on \mathbb{C}^1 given by $t = 0$. Then \bar{X} is smooth, the surface X has one singular point of type \mathbb{D}_5 . Put $Z = \mathbb{C}^1$. Then the map

$$(x, y, z, w) \longrightarrow (t^2x, ty, z, t^4w)$$

induces a birational map $\rho: V \dashrightarrow \bar{V}$ such that the diagrams 5.1 and isomorphism 5.2 exist, and ρ is not biregular. But $\text{lct}(X) = 1/4$ and $\text{lct}(\bar{X}) = 2/3$ (see Example 1.7 and Lemma 3.7).

The following result holds (see Examples 1.7 and 5.4).

Theorem 5.7. *The map ρ is an isomorphism if one of the following conditions hold:*

- *the varieties X and \bar{X} have log terminal singularities, and $\text{lct}(X) + \text{lct}(\bar{X}) > 1$;*
- *the variety X has log terminal singularities, and $\text{lct}(X) \geq 1$.*

Proof. Suppose that the variety X has log terminal singularities, the inequality $\text{lct}(X) \geq 1$ holds, and ρ is not an isomorphism. Let D be a general very ample divisor on Z . Put

$$\Lambda = |-nK_V + \pi^*(nD)|, \quad \Gamma = |-nK_{\bar{V}} + \bar{\pi}^*(nD)|, \quad \bar{\Lambda} = \rho(\Lambda), \quad \bar{\Gamma} = \rho^{-1}(\Gamma),$$

where n is a natural number such that Λ and Γ have no base points. Put

$$M_V = \frac{2\varepsilon}{n} \Lambda + \frac{1-\varepsilon}{n} \Gamma, \quad M_{\bar{V}} = \frac{2\varepsilon}{n} \bar{\Lambda} + \frac{1-\varepsilon}{n} \bar{\Gamma},$$

where ε is a positive rational number.

The log pairs (V, M_V) and $(\bar{V}, M_{\bar{V}})$ are birationally equivalent, and $K_V + M_V$ and $K_{\bar{V}} + M_{\bar{V}}$ are ample. The uniqueness of canonical model (see Theorem 1.3.20 in [3]) implies that ρ is biregular if the singularities of both log pairs (V, M_V) and $(\bar{V}, M_{\bar{V}})$ are canonical.

The linear system Γ does not have base points. Thus, there is a rational number ε such that the log pair $(\bar{V}, M_{\bar{V}})$ is canonical. So, the log pair (V, M_V) is not canonical. Then the log pair

$$\left(V, X + \frac{1-\varepsilon}{n} \bar{\Gamma}\right)$$

is not log canonical, because Λ does not have base points, and $\bar{\Gamma}$ does not have base points outside of the fiber X , which is a Cartier divisor on the variety V . The log pair

$$\left(X, \frac{1-\varepsilon}{n} \bar{\Gamma}|_X\right)$$

is not log canonical by Theorem 17.6 in [12], which is impossible, because $\text{lct}(X) \geq 1$.

To conclude the proof we may assume that the varieties X and \bar{X} have log terminal singularities, the inequality $\text{lct}(X) + \text{lct}(\bar{X}) > 1$ holds, and ρ is not an isomorphism.

Let $\Lambda, \Gamma, \bar{\Lambda}, \bar{\Gamma}$ and n be the same as in the previous case. Put

$$M_V = \frac{\text{lct}(\bar{X}) - \varepsilon}{n} \Lambda + \frac{\text{lct}(X) - \varepsilon}{n} \Gamma, \quad M_{\bar{V}} = \frac{\text{lct}(\bar{X}) - \varepsilon}{n} \bar{\Lambda} + \frac{\text{lct}(X) - \varepsilon}{n} \bar{\Gamma},$$

where ε is a sufficiently small positive rational number. Then it follows from the uniqueness of canonical model that ρ is biregular if both log pair (V, M_V) and $(\bar{V}, M_{\bar{V}})$ are canonical.

Without loss of generality, we may assume that the singularities of the log pair (V, M_V) are not canonical. Arguing as in the previous case, we see that the log pair

$$\left(X, \frac{\text{lct}(X) - \varepsilon}{n} \bar{\Gamma}|_X\right)$$

is not log canonical, which is impossible, because $\bar{\Gamma}|_X \equiv -nK_X$. □

The assertion of Theorem 5.7 can not be improved (see Examples 5.3, 5.5, 5.6).

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